Griffiths inequalities for the Gaussian spin glass

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Abstract. The Griffiths inequalities for Ising spin-glass models with Gaussian randomness of non-vanishing mean are proved using properties of the Gaussian distribution and gauge symmetry of the system. These inequalities imply that correlation functions are non-negative and monotonic along the Nishimori line in the phase diagram. From this result, the existence of thermodynamic limit for correlation functions and pressure is proved under free and fixed boundary conditions. Relations between the location of multicritical points are also derived for different lattices.

1. Introduction

Two Griffiths inequalities provide a significant insight about the phase transitions in ferromagnetic Ising models [1]. One of their formulations states that if the sets of Ising spins $S_A = \prod_{i \in A} S_i$ are coupled by an energy $-J_A S_A$ (for some positive J_A), the free energy F and all the correlations are monotonic functions of the strength of any interactions J's, namely:

$$- \frac{\mathrm{d}}{\mathrm{d}J_A} F \geq 0 \ , \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}J_B} \langle S_A \rangle \geq 0 \ . \tag{2}$$

These inequalities can be used in ferromagnetic Ising models to prove that the free energy and correlation functions have the thermodynamic limit under several boundary conditions and to demonstrate the existence of phase transitions for various lattices. The proof of equations (1) and (2) [2] assumes that all the interactions among spins are ferromagnetic, a condition that clearly fails in spin glass models which have both ferromagnetic and antiferromagnetic interactions.

Attempts to extend even only partially those or similar inequalities to the spin glasses have been unsuccessful until very recently after the technique of integration by parts has been powerfully exploited in the mathematically rigorous approaches: in [3, 4] the main results are based on correlation inequalities that represents the equivalent of the *first* Griffiths inequality (1) for the Sherrington-Kirkpatrick model and the Edwards-Anderson model respectively. The monotonicity properties in [3, 4] are proved not with

respect to the strength of the interaction but with respect to the *variance* of the random interaction.

In this paper we show that a Gaussian Ising spin-glass model does fulfil the *first* and the *second* Griffiths inequalities (1) and (2) with respect to the *mean* of the distribution: pressure and correlations are monotonic functions with respect to the mean. The results are proved on the Nishimori line (NL), a restricted space of the phase diagram in which several exact results follow from the gauge symmetries of the system [6, 7].

We apply the resulting inequalities to prove the existence of the thermodynamic limit for the pressure and correlation functions under free and fixed boundary conditions. Moreover, we derive inequalities on the location of multicritical points.

In the next section, we present our results and their proofs. The applications of the inequalities are discussed in the last section. Some of the details of calculations are described in the appendix.

2. Inequalities

Let us consider a finite box Ω, subset of a regular lattice $\mathcal{L}$. To each point $i \in \Omega$ we associate an Ising spin ($S_i = \pm 1$) and denote their products by

$$S_A = \prod_{i \in A} S_i \ . \tag{3}$$

We consider the spin glass model defined by the random potential

$$U_\Omega = \sum_{A \subset \Omega} \beta_A J_A S_A \ , \tag{4}$$

where $\beta_A \geq 0$ is the inverse of local temperature of subset A and J_A is a quenched random variable which follows the Gaussian distribution with positive mean $[J_A] = J_{A0}$ and variance $[(J_A - J_{A0})^2] = \sigma_A^2$. The *pressure* function is defined as

$$P = [\log \sum_S e^{U_\Omega}] \ , \tag{5}$$

from which all physical quantities can be derived.

The NL is defined in terms of the parameters $x_A \geq 0$ (for all the A) as [6, 7]

$$\beta_A = \frac{x_A}{\sigma_A} \ , \quad J_{A0} = \sigma_A x_A \ . \tag{6}$$

Our results are the following inequalities:

$$\frac{\mathrm{d}P}{\mathrm{d}x_B} = x_B[\langle S_B + 1 \rangle] \geq 0 \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}x_B}[\langle S_C \rangle] = 2x_B[(\langle S_B S_C \rangle - \langle S_B \rangle \langle S_C \rangle)^2] \geq 0 \ . \tag{8}$$

Both inequalities hold for arbitrary subsets B, C as long as the parameters satisfy NL condition (6).

The first and second inequalities are proved using some properties of the Gaussian distribution and the gauge theory. To prove the first inequality (7), we observe that on the NL the formula of the *total derivative* gives

$$\frac{\mathrm{d}P}{\mathrm{d}x_B} = \frac{\partial P}{\partial \beta_B}\frac{\mathrm{d}\beta_B}{\mathrm{d}x_B} + \frac{\partial P}{\partial J_{B0}}\frac{\mathrm{d}J_{B0}}{\mathrm{d}x_B} = \frac{1}{\sigma_B}\frac{\partial P}{\partial \beta_B}\bigg|_{\mathrm{NL}} + \sigma_B\frac{\partial P}{\partial J_{B0}}\bigg|_{\mathrm{NL}} . \tag{9}$$

As shown in the appendix, we can derive the following identities from the properties of the Gaussian distribution for any operator O:

$$[J_B O] = J_{B0}[O] + \sigma_B^2\left[\frac{\partial O}{\partial J_B}\right] \tag{10}$$

$$\frac{\partial}{\partial J_{B0}}[O] = \left[\frac{\partial O}{\partial J_B}\right] . \tag{11}$$

Note that both equations (10) and (11) are valid on and away from the NL. The identity (11) yields

$$\frac{\partial P}{\partial J_{B0}} = \beta_B[\langle S_B\rangle] . \tag{12}$$

and a direct computation provides

$$\frac{\partial P}{\partial \beta_B} = [J_B\langle S_B\rangle] . \tag{13}$$

Using the gauge theory we have on the NL [6, 7]

$$[\langle J_B S_B\rangle] = J_{B0} = \sigma_B x_B . \tag{14}$$

From equations (9), (12)-(14), we immediately find equation (7).

It is also possible to obtain a related inequality, an explicit bound for the correlation function, as

$$[\langle S_B\rangle] \geq \frac{x_B^2}{1 + x_B^2} \quad (B \in \{A\}) , \qquad [\langle S_B\rangle] \geq 0 \quad (B \notin \{A\}) . \tag{15}$$

For this purpose we observe that, when $B \in \{A\}$,

$$[\langle J_B S_B\rangle] = \int \prod_{A\subset\Omega} \mathrm{d}J_A \left(\sqrt{P_A(J_A)}J_B\right)\left(\sqrt{P_A(J_A)}\langle S_B\rangle\right) . \tag{16}$$

Let us square both sides of the above equation and apply the Cauchy-Schwarz inequality to obtain

$$[\langle J_B S_B\rangle]^2 \leq (\sigma_B^2 + J_{B0}^2)[\langle S_B\rangle^2] . \tag{17}$$

Since the gauge theory yields an identity [6, 7]

$$[\langle S_B\rangle] = [\langle S_B\rangle^2] \tag{18}$$

on the NL under certain boundary conditions (free, periodic or fixed (all spins up)), we obtain equation (15) from equations (14) and (17). When $B \notin \{A\}$, the inequality (15) is a direct consequence of the identity (18). We note that the Gaussian distribution is not essential for the proof of the above inequality (15).

The proof of the second inequality (8) is similarly carried out. Derivative by the parameter x_B is expressed as in equation (9) by

$$\frac{\mathrm{d}}{\mathrm{d}x_B}[\langle S_C\rangle] = \frac{1}{\sigma_B}\frac{\partial}{\partial \beta_B}[\langle S_C\rangle]\bigg|_{\mathrm{NL}} + \sigma_B\frac{\partial}{\partial J_{B0}}[\langle S_C\rangle]\bigg|_{\mathrm{NL}} . \tag{19}$$

Derivative by β_B is easily calculated as

$$\frac{\partial}{\partial \beta_B}[\langle S_C\rangle] = [J_B\langle S_B S_C\rangle] - [J_B\langle S_B\rangle\langle S_C\rangle] . \tag{20}$$

Substitution of equation (10) into equation (20) yields

$$\begin{aligned}\frac{1}{\sigma_B}\frac{\partial}{\partial \beta_B}[\langle S_C\rangle] = {} & \frac{J_{B0}}{\sigma_B}[\langle S_B S_C\rangle - \langle S_B\rangle\langle S_C\rangle] \\ & - 2\sigma_B\beta_B[\langle S_B\rangle\langle S_B S_C\rangle - \langle S_B\rangle^2\langle S_C\rangle] ,\end{aligned} \tag{21}$$

and the identity (11) gives

$$\sigma_B\frac{\partial}{\partial J_{B0}}[\langle S_C\rangle] = \sigma_B\beta_B[\langle S_B S_C\rangle - \langle S_B\rangle\langle S_C\rangle] . \tag{22}$$

From equation (19) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x_B}[\langle S_C\rangle] = 2x_B[\langle S_B S_C\rangle - \langle S_B\rangle\langle S_C\rangle - \langle S_B\rangle\langle S_B S_C\rangle + \langle S_B\rangle^2\langle S_C\rangle] . \tag{23}$$

The gauge theory yields the following identities on the NL under the same boundary conditions as the identity (18) [6, 7]:

$$\begin{aligned}[\langle S_B S_C\rangle] &= [\langle S_B S_C\rangle^2] \\ [\langle S_B\rangle\langle S_C\rangle] &= [\langle S_B\rangle\langle S_B S_C\rangle] = [\langle S_B\rangle\langle S_C\rangle\langle S_B S_C\rangle] \\ [\langle S_B\rangle^2\langle S_C\rangle] &= [\langle S_B\rangle^2\langle S_C\rangle^2] .\end{aligned} \tag{24}$$

Therefore, we obtain from equation (23)

$$\frac{\mathrm{d}}{\mathrm{d}x_B}[\langle S_C\rangle] = 2x_B[(\langle S_B S_C\rangle - \langle S_B\rangle\langle S_C\rangle)^2] \geq 0 . \tag{25}$$

In general, the following relation is proved similarly:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x_B}[\langle S_C\rangle^{2k-1}] &= \frac{\mathrm{d}}{\mathrm{d}x_B}[\langle S_C\rangle^{2k}] \\ &= 2k(2k-1)x_B[\langle S_B\rangle^{2k-2}(\langle S_B S_C\rangle - \langle S_B\rangle\langle S_C\rangle)^2] \geq 0 .\end{aligned} \tag{26}$$

It is also possible to prove concavity of the pressure from the derivative of equation (7), since the second derivative of the pressure is calculated as

$$\frac{\mathrm{d}^2 P}{\mathrm{d}x_B \mathrm{d}x_C} = \begin{cases} x_B\dfrac{\mathrm{d}}{\mathrm{d}x_C}[\langle S_B\rangle] \geq 0 & (B \neq C) \\[2mm] [\langle S_B\rangle] + 1 + x_B\dfrac{\mathrm{d}}{\mathrm{d}x_B}[\langle S_B\rangle] \geq 0 & (B = C) \end{cases} \tag{27}$$

where we use equations (8) and (15).

3. Discussions on physical consequences

The inequality (15) implies that a correlation function $[\langle S_B \rangle]$ is non-negative and increases toward unity when the parameter x_B of the corresponding subset B tends to infinity. This result is reasonable: for large x_B, the interaction J_B is almost ferromagnetic and the local temperature is nearly zero, and therefore all the spins in the subset B become parallel to each other.

An immediate result from the second inequality (8) is that an arbitrary n-point correlation function, including order parameters, is an increasing function of a parameter for any subset. In particular, a two-point correlation function increases with x (which is proportional to the inverse temperature and the centre of distribution) and thus the correlation length becomes larger as x increases, a natural result.

The two inequalities have profound consequences on the structure and existence of the thermodynamic limit $\Omega \nearrow \mathcal{L}$ for both pressure and correlation functions. The first inequality (7) tells us that the pressure is monotonically increasing with any x_A. Since on the NL the Boltzmann weight admits the representation

$$
e^{\sum_{A \in \Omega} x_A j_A S_A} , \tag{28}
$$

where $j_A = J_A / \sigma_A$ is a Gaussian variable with mean $[j_A] = x_A \geq 0$ and variance $[(j_A - x_A)^2] = 1$, the pressure can be expressed as

$$
P = [\log \sum_S e^{\sum_{A \in \Omega} x_A j_A S_A}] , \tag{29}
$$

and the correlation as

$$
[\langle S_C \rangle] = \left[\frac{\sum_S S_C e^{\sum_{A \in \Omega} x_A j_A S_A}}{\sum_S e^{\sum_{A \in \Omega} x_A j_A S_A}} \right] , \tag{30}
$$

where both P and $[\langle S_C \rangle]$ are functions only of x's. From equation (29) we see that the x_A *tune* the interaction of the set of spins A.

We can show the existence of the thermodynamic limit for the pressure per spin under free boundary condition. Take a large cube Ω (consider for simplicity $d = 2$) and cut it in four identical cubes Ω_i, $i = 1, ..., 4$. We can tune *off* the interactions among the cubes simply by setting all the x's among them equal to zero. In this case the total pressure is $P_{\Omega_1} + P_{\Omega_2} + P_{\Omega_3} + P_{\Omega_4}$. Then we can tune *on* the interactions from zero to their original value and the total pressure becomes P_Ω. The monotonicity property (7) gives

$$
P_{\Omega_1} + P_{\Omega_2} + P_{\Omega_3} + P_{\Omega_4} \leq P_\Omega , \tag{31}
$$

which is the well known sub-additivity property for the pressure and implies the existence of its density in the thermodynamic limit under two natural assumptions of invariance by translation of the Gaussian distributions and the stability boundedness [5].

Moreover the second inequality (8) can be used to prove the existence of the thermodynamic limit for the correlation functions in the same way as the first is used to prove the existence of the pressure. In fact, from equation (28), we see that when $x_A = 0$

there is no interaction among the subset A, and to increase x_A from zero is equivalent to add an interaction to the subset A. From the second inequality (8), addition of an interaction for any subset increases all correlation functions.

This result can be used for the proof that correlation functions have a thermodynamic limit under free boundary conditions. Let us consider two finite sets $\Omega' \subset \Omega$. The subset Ω is obtained from Ω' by adding interactions. Then it is clear from the previous argument that

$$[\langle S_B \rangle]^{(\text{free})}_{\Omega'} \leq [\langle S_B \rangle]^{(\text{free})}_{\Omega}. \tag{32}$$

This implies that the correlation functions monotonically increase with the system size. Since correlation functions are bounded by unity, each of them tends to its unique limit as the volume of Ω tends to infinity. Therefore the thermodynamic limit for correlation functions exists on the NL.

Monotonicity of the correlation functions with system size can also be proved under fixed boundary condition. Fixed boundary conditions ($S_i = +1$ for all boundary sites $i \in \partial\Omega$) is represented by applying very strong positive magnetic fields to all sites outside Ω. This is equivalent to $x_A \to \infty$ for $A = \{i\}$, $i \in \partial\Omega$. Since the set $\Omega \supset \Omega'$ is obtained from Ω' by reducing magnetic fields for sites $i \in \Omega \setminus \Omega'$, we have

$$[\langle S_B \rangle]^{(\text{fix})}_{\Omega'} \geq [\langle S_B \rangle]^{(\text{fix})}_{\Omega} \tag{33}$$

from the inequality (8). Because the inequality (15) yields lower bounds for $[\langle S_B \rangle]$, there should exist a well-defined limit as $\Omega \nearrow \mathcal{L}$. The thermodynamic limit of $[\langle S_B \rangle]^{(\text{fix})}$ may not necessarily be equal to that of $[\langle S_B \rangle]^{(\text{free})}$, but the former cannot be less than the latter.

The second inequality can also be used for the argument about the location of the multicritical points, which are believed to lie on the NL [6, 7], for various lattices. For example, let us consider three two-dimensional lattices, the triangular (TR), square (SQ) and hexagonal (HEX) lattices. The triangular lattice is obtained from the square lattice with addition of bonds, and the square lattice is obtained from the hexagonal. Thus, the magnetizations of three lattices satisfy the following relation:

$$[\langle S_i \rangle]_{\text{HEX}} \leq [\langle S_i \rangle]_{\text{SQ}} \leq [\langle S_i \rangle]_{\text{TR}} . \tag{34}$$

Since the multicritical temperature is defined as

$$T_c = \sup\{T; [\langle S_i \rangle] > 0\} , \tag{35}$$

we obtain

$$T_c^{\text{HEX}} \leq T_c^{\text{SQ}} \leq T_c^{\text{TR}} . \tag{36}$$

Similarly, the multicritical temperature of the simple cubic lattice is higher then that of the square lattice because the former lattice is constructed from the latter by adding interactions.

It is an interesting future problem to prove our results for other models, for example, the $\pm J$ Ising model, and to develop similar analyses away from the NL.

After submission of the manuscript we learnt that Kitatani [8] had discussed a related problem.

Acknowledgments

Appendix

In this appendix, we derive equations (10) and (11) using some properties of the Gaussian distribution

$$P_A(J_A) = \frac{1}{\sqrt{2\pi\sigma_A^2}} \exp\left(-\frac{(J_A - J_{A0})^2}{2\sigma_A^2}\right) . \tag{A.1}$$

Let us consider the following quantity,

$$\left[\frac{\partial}{\partial J_B}O(\{J_A\})\right] = \int \prod_{A\subset\Omega} dJ_A P_A(J_A)\frac{\partial}{\partial J_B}O(\{J_A\}) , \tag{A.2}$$

where $O(\{J_A\})$ is a function of interactions $\{J_A\}$. Since Gaussian distribution $P_A(J_A)$ decays rapidly as $|J_A| \to \infty$, we may rewrite the right-hand side using integration by parts as

$$\left[\frac{\partial}{\partial J_B}O(\{J_A\})\right] = \int \prod_{\substack{A\subset\Omega\\A\neq B}} dJ_A P_A(J_A) dJ_B P_B(J_B)\frac{\partial}{\partial J_B}O(\{J_A\}) \tag{A.3}$$

$$= \int \prod_{\substack{A\subset\Omega\\A\neq B}} dJ_A P_A(J_A) dJ_B\left(-\frac{\partial}{\partial J_B}P_B(J_B)\right)O(\{J_A\}) . \tag{A.4}$$

Calculating the derivative of the Gaussian yields

$$\frac{\partial}{\partial J_B}P_B(J_B) = \frac{J_{B0} - J_B}{\sigma_B^2}P_B(J_B) . \tag{A.5}$$

This relation shows that

$$\left[\frac{\partial}{\partial J_B}O(\{J_A\})\right] = \int \prod_{A\subset\Omega} dJ_A P_A(J_A)\frac{J_B - J_{B0}}{\sigma_B^2}O(\{J_A\}) \tag{A.6}$$

$$= \frac{1}{\sigma_B^2}\left\{[J_B O(\{J_A\})] - J_{B0}[O(\{J_A\})]\right\} \tag{A.7}$$

which is equation (10).

For the proof of (11), we use the property that the Gaussian distribution $P_B(J_B)$ is a function of $J_B - J_{B0}$,

$$\frac{\partial}{\partial J_B}P_B(J_B) = -\frac{\partial}{\partial J_{B0}}P_B(J_B) . \tag{A.8}$$

Substitution of (A.8) into (A.4) yields

$$
\begin{aligned}
\left[\frac{\partial}{\partial J_B} O(\{J_A\})\right] &= \int \prod_{\substack{A \subseteq \Omega \\ A \neq B}} \mathrm{d}J_A P_A(J_A) \mathrm{d}J_B \left(\frac{\partial}{\partial J_{B0}} P_B(J_B)\right) O(\{J_A\}) \\
&= \frac{\partial}{\partial J_{B0}} \int \prod_{A \subseteq \Omega} \mathrm{d}J_A P_A(J_A) O(\{J_A\}) \\
&= \frac{\partial}{\partial J_{B0}} [O(\{J_A\})] \, .
\end{aligned}
\tag{A.9}
$$

References

[1] Griffiths R B 1972 *Phase Transitions and Critical Phenomena* vol 1 ed, C Domb and M S Green (London: Academic) p 7
[2] Griffiths R B 1967 *J. Math. Phys* **8** 478, 484
[3] Guerra F and Toninelli F 2002 *Commun. Math. Phys.* **230** 71
[4] Contucci P and Graffi S 2003 *Preprint* math-ph/0306006 (*Commun. Math. Phys* at press)
[5] Contucci P and Graffi S 2004 *J. Stat. Phy.* **115** 581
[6] Nishimori H 1981 *Prog. Theor. Phys,* **66** 1169
[7] Nishimori H 2001 *Statistical Physics of Spin Glasses and Information Processing: An Introduction* (Oxford: Oxford University Press)
[8] Kitatani H 1994 *J. Phys. Soc. Jpn.* **63** 2070